Topic 11 – Review of power series

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<u>Def:</u> An <u>infinite sum</u> is a sum of the form

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

[This sum starts at $n=0$, but you can start n at any number]

We say that the above sum <u>converges</u> to S and write $\sum_{n=0}^{\infty} a_n = S$ if

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N} a_n\right] = S$$

$$\lim_{N\to\infty}\left[\underbrace{a_0 + a_1 + \cdots + a_N}\right] = S$$

called partial sums

If no such S exists, then the series diverges.

Ex: Consider

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

Let's calculate some partial sums.

N	$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^N}$
1	$\frac{1}{2}=0{,}5$
2	$\frac{1}{2}+\frac{1}{2^2}=0{,}75$
3	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=0.875$
4	$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=0.9375$
5	$0{,}96875$
$\vdots$	$\vdots$
50	$0{,}\underbrace{999\ldots99}_{15\ 9\text{'s}}11182$
$\vdots$	$\vdots$
100	$0{,}\underbrace{999\ldots99}11139\ldots$

30 9's

In Calc II (Math 2120) you show that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \longleftarrow S = 1 \text{ from def}$$

<u>Def:</u> A <u>power series</u> is an infinite sum of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

Where $x_0, a_0, a_1, a_2, \dots$ are constants and x is a variable.

We say the power series is <u>centered</u> at x_0.

<u>Ex:</u> (Geometric sum)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$= \underset{a_0=1}{1} + \underset{a_1=1}{1} \cdot (x - \underset{x_0=0}{0}) + \underset{a_2=1}{1} \cdot (x - \underset{x_0=0}{0})^2 + \cdots$$

$x_0 = 0$ is center

Ex:

$$\sum_{n=0}^{\infty} \frac{1}{2^n}(x-3)^n$$

$$= 1 + \frac{1}{2}(x-3) + \frac{1}{2^2}(x-3)^2 + \cdots$$

The center is $x_0 = 3$

Ex: (Back to geometric series...)

In Calculus you show that

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{1+x+x^2+x^3+\cdots} = \frac{1}{1-x} \quad \text{if } -1 < x < 1$$

The sum diverges otherwise.

For example,

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{2}\right)^3 + \cdots$$

$$= \frac{1}{1 - 3/4} = \frac{1}{1/4} = 4$$

$x = 3/4$

$-1 < x < 1$

And

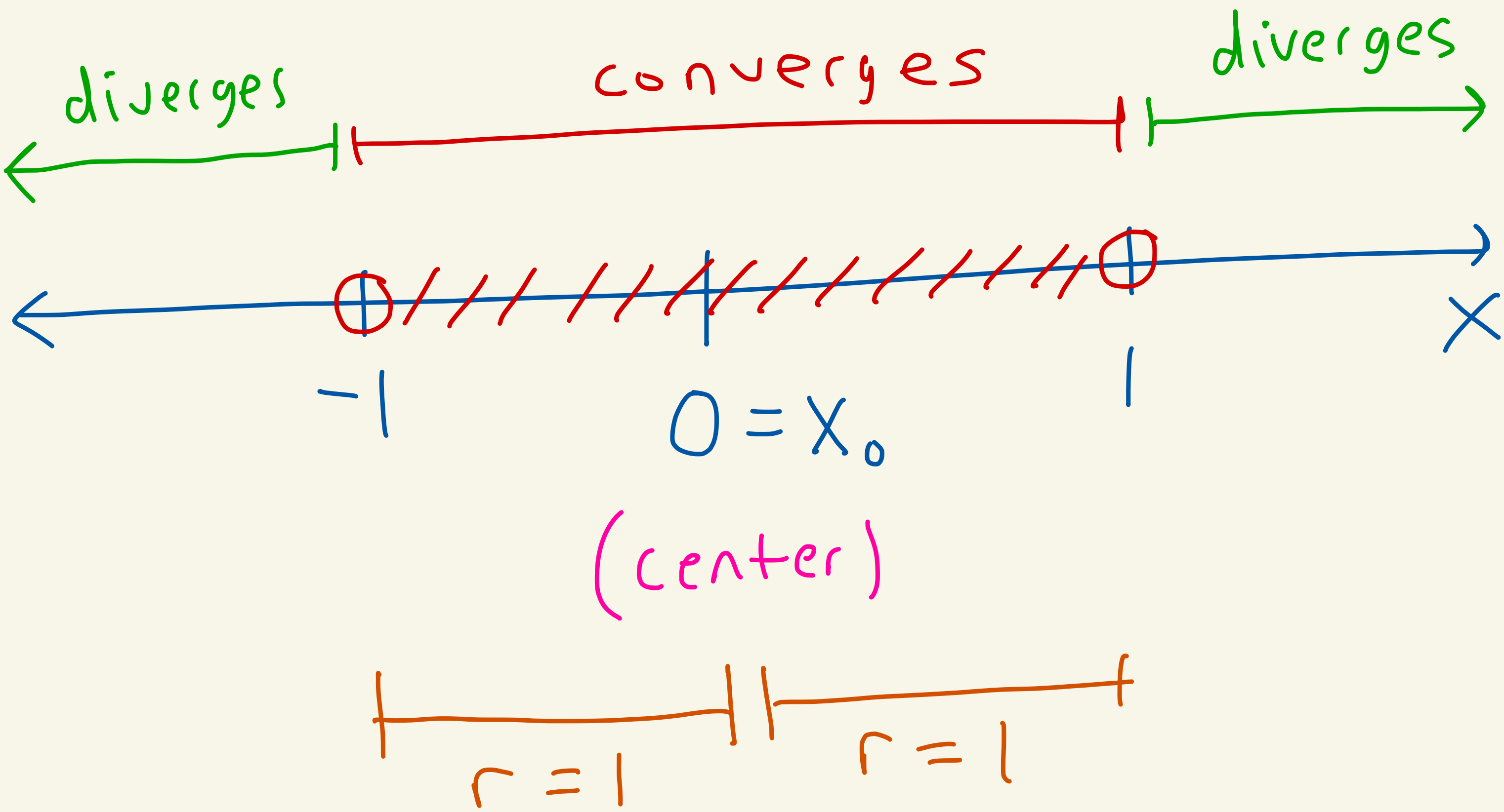
$$\sum_{n=0}^{\infty} \pi^n = 1 + \pi + \pi^2 + \pi^3 + \cdots$$

diverges to ∞ since $x = \pi$ does not satisfy $-1 < x < 1$.

If we write

$$f(x) = \sum_{n=0}^{\infty} x^n$$

then we can plug in $-1 < x < 1$ but not other x's. In a picture we get:



$r = 1$ is called the radius of convergence.

Ex: Recall from Calculus:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

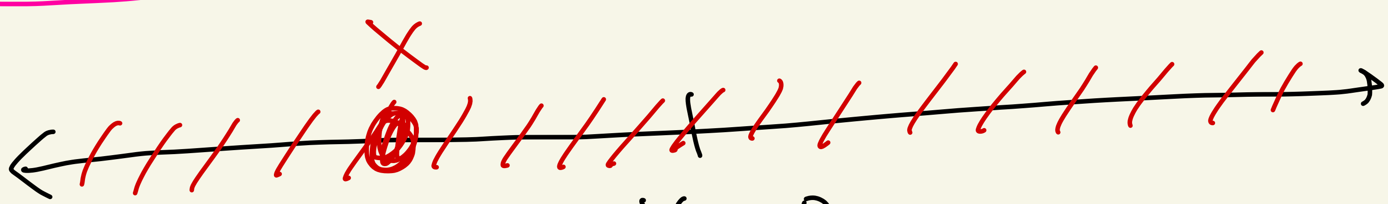
$$= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$\vdots$

This series converges for every x.

The center is $x_0 = 0$.



$x_0 = 0$

You can plug any x into sum.

The radius of convergence is $r = \infty$.

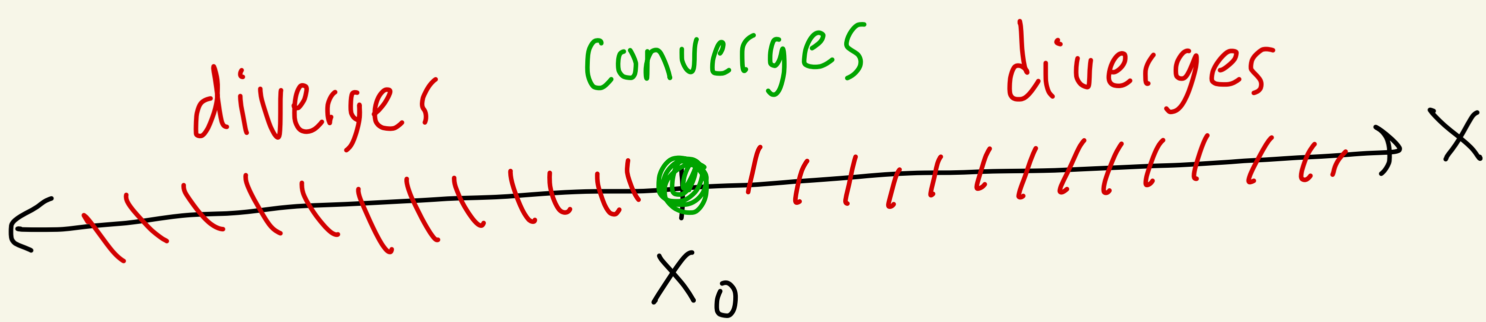
Radius of convergence facts:

There are three possibilities for a power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

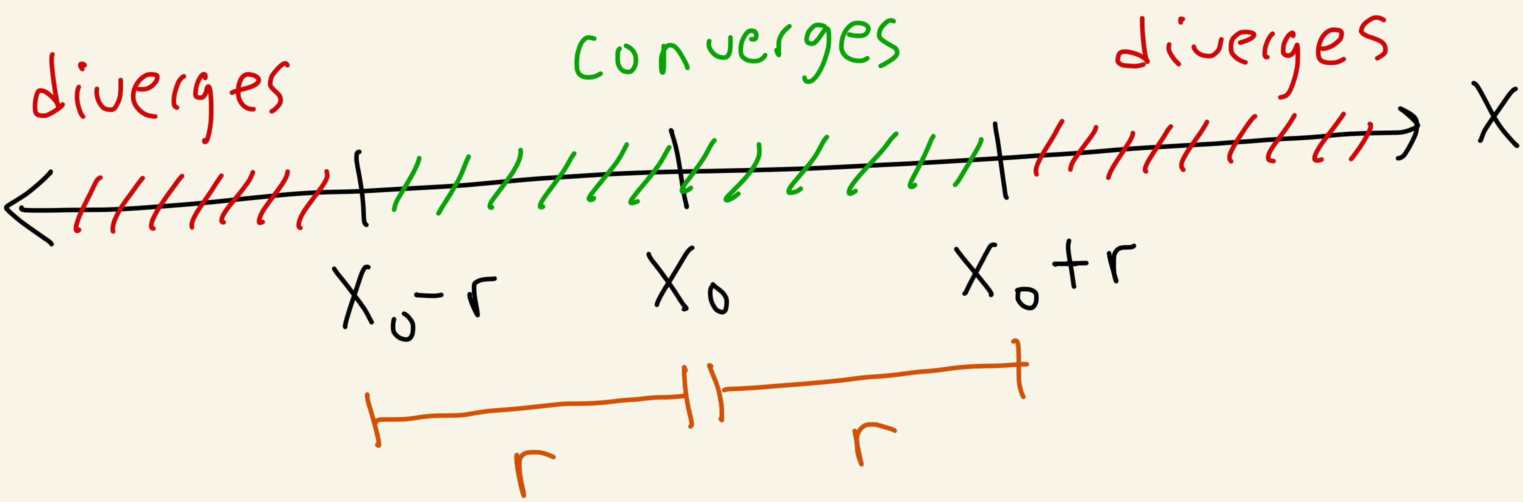
centered at x_0.

① The series only converges when $x = x_0$



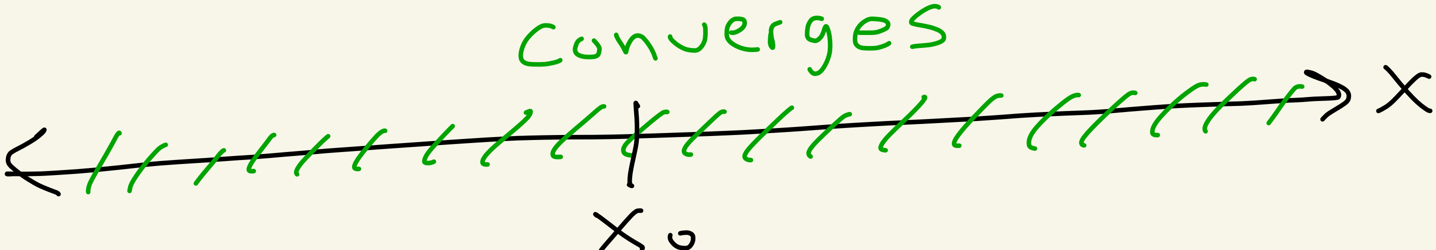
$r = 0$ is the radius of convergence

(2) there exists $r > 0$ where the series converges for all x with $x_0 - r < x < x_0 + r$



It can diverge or converge at the endpoints $x_0 - r$ and $x_0 + r$.
r is called the radius of convergence

(3) the series converges for all x.
$r = \infty$ is the radius of convergence.



Ex: Recall

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

These both converge for all x.
radius of convergence is $r = \infty$
Both are centered at $x_0 = 0$.

So, for example,

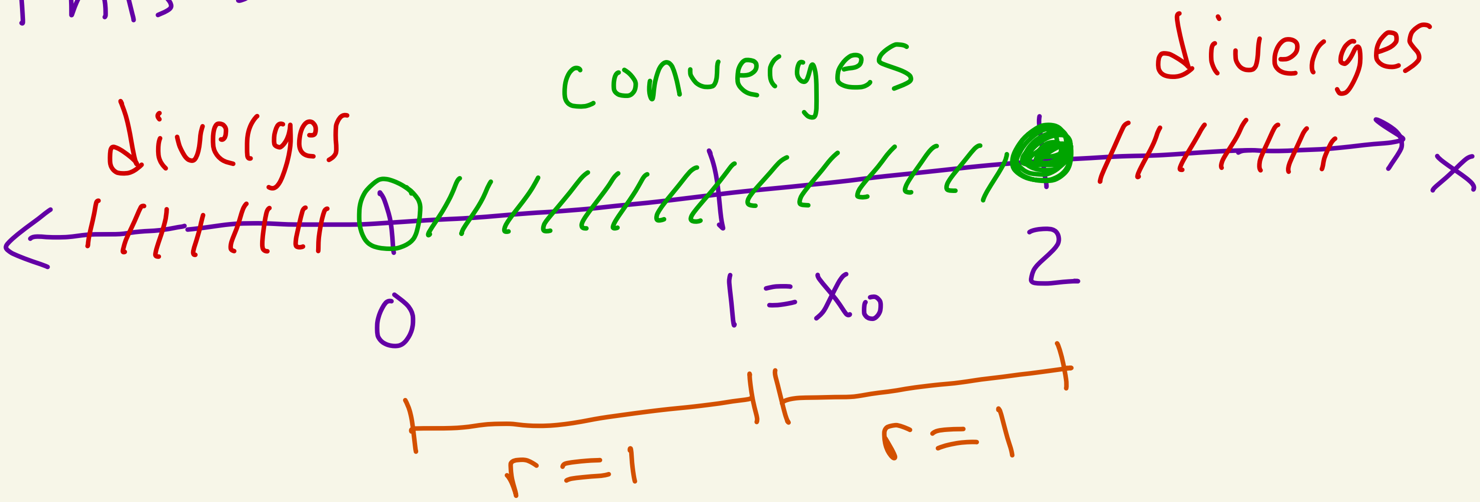
$$\sin(2) = 2 - \frac{1}{3!}(2)^3 + \frac{1}{5!}(2)^5 - \frac{1}{7!}(2)^7 + \dots$$

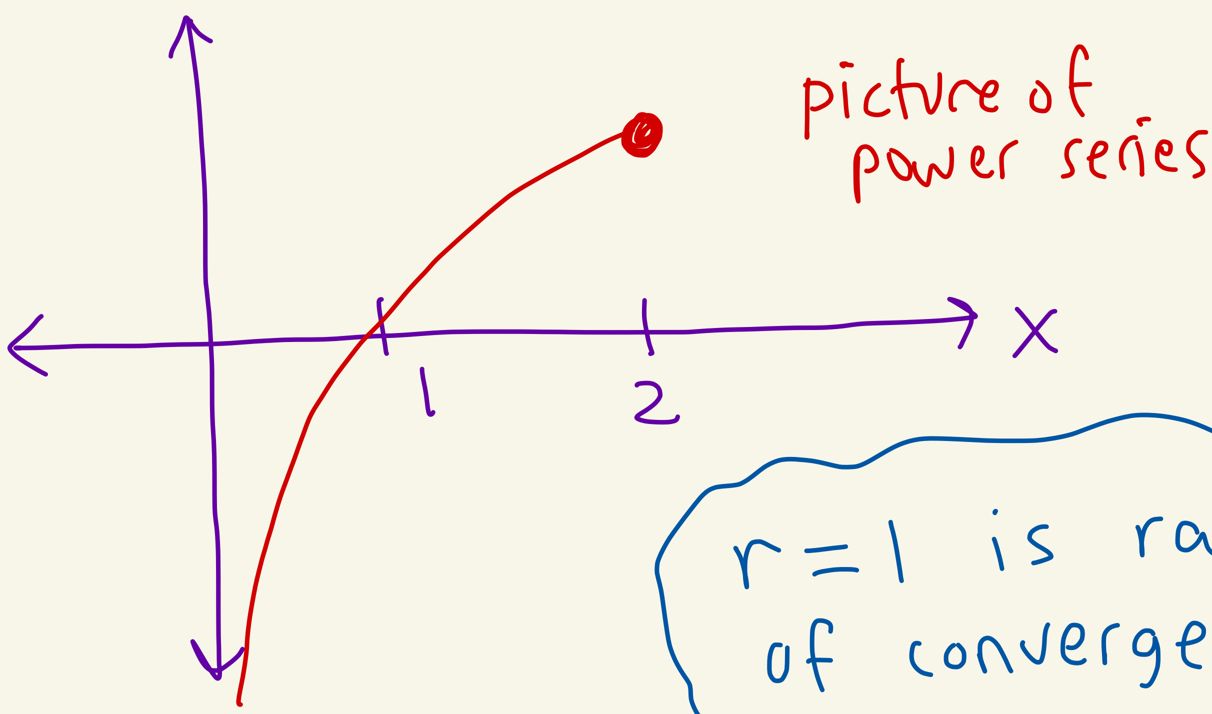
Ex: Recall

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \leftarrow \boxed{x_0 = 1}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

This series converges for $0 < x \leq 2$.



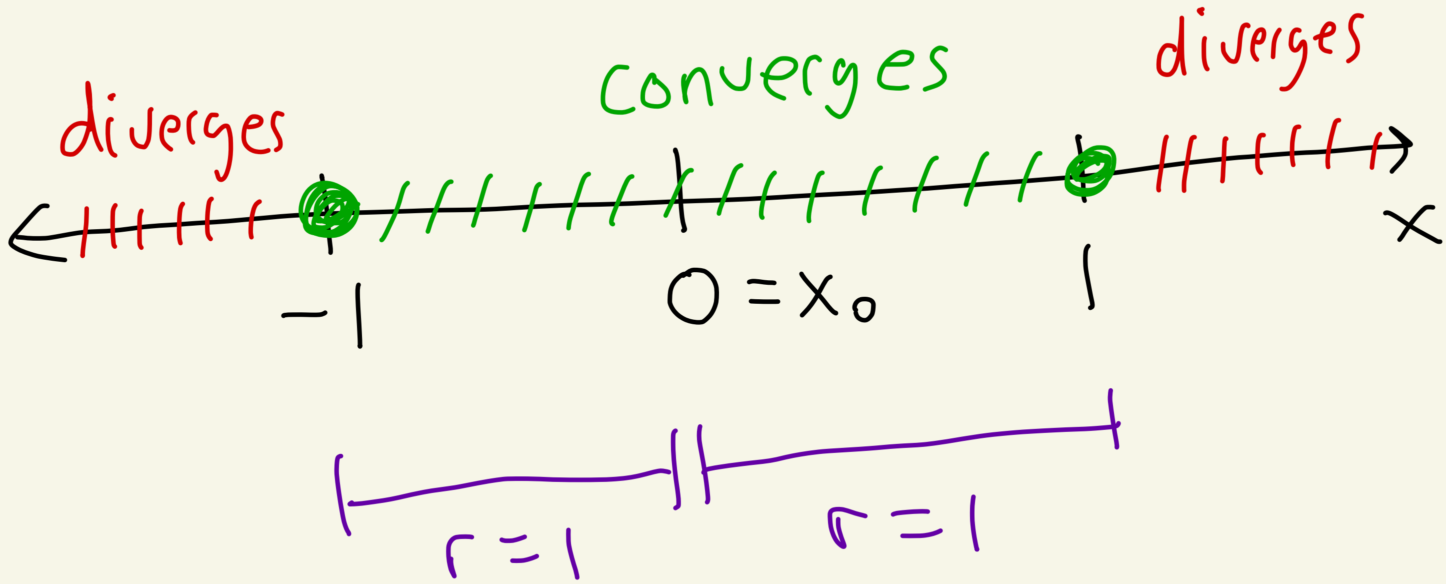


r=1 is radius of convergence

Ex: If $-1 \leq x \leq 1$, then

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$



$r=1$ is radius of convergence

Fun fact: Can use the above to approximate π.

$$\frac{\pi}{4} = \tan^{-1}(1) = \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots}_{\text{plug } x=1 \text{ into power series above}}$$

Thus,

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

Theorem: (Taylor series)

If
$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

has radius of convergence $r > 0$.

Then,
$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

So,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

Ex: Find the power series for $f(x) = x^2$ centered at $x_0 = 2$.

$\boxed{\text{plug in center } x_0 = 2}$

$f(x) = x^2 \longleftarrow f(2) = 4$

$f'(x) = 2x \longleftarrow f'(2) = 4$

$f''(x) = 2 \longleftarrow f''(2) = 2$

$f'''(x) = 0 \longleftarrow f'''(2) = 0$

$f''''(x) = 0 \longleftarrow f''''(2) = 0$

$\vdots \quad \vdots \qquad \vdots \quad \vdots$

all 0

all 0

$$\underbrace{x^2}_{f(x)} = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2$$

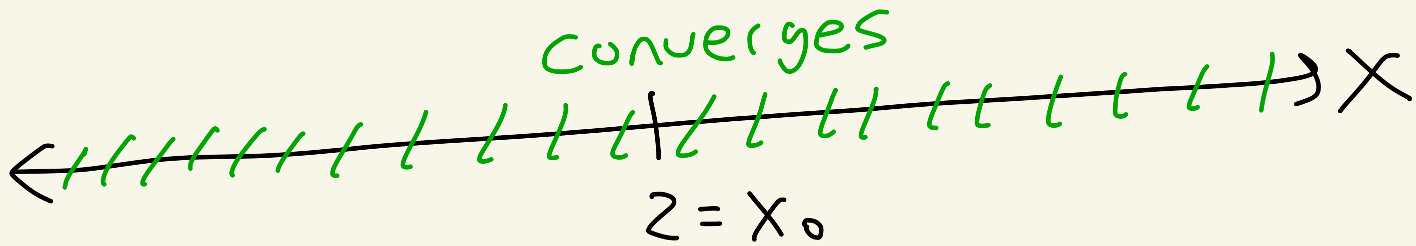
$$+ \underbrace{\boxed{\frac{f'''(2)}{6}(x-2)^3 + \cdots}}_{\text{zero}}$$

$$= 4 + 4(x-2) + \frac{2}{2}(x-2)^2$$

So,

$$\boxed{x^2 = 4 + 4(x-2) + (x-2)^2}$$

This will converge for every x since its just a finite sum.

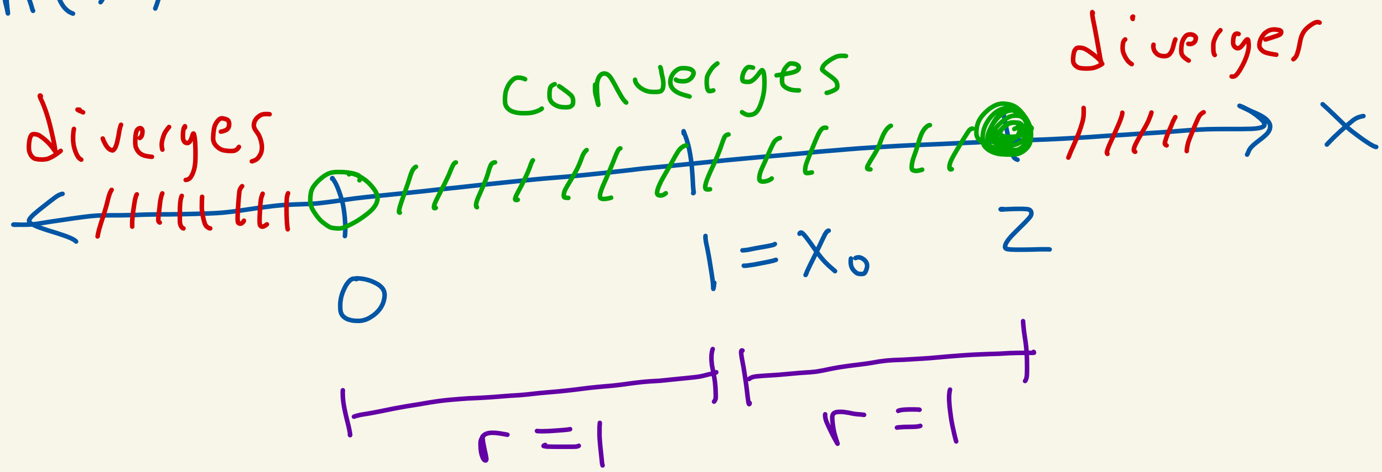


radius of convergence is $r = \infty$.

<u>Fact</u>: You can differentiate or integrate a function by differentiating or integrating it's power series term by term. This process doesn't change the radius of convergence.

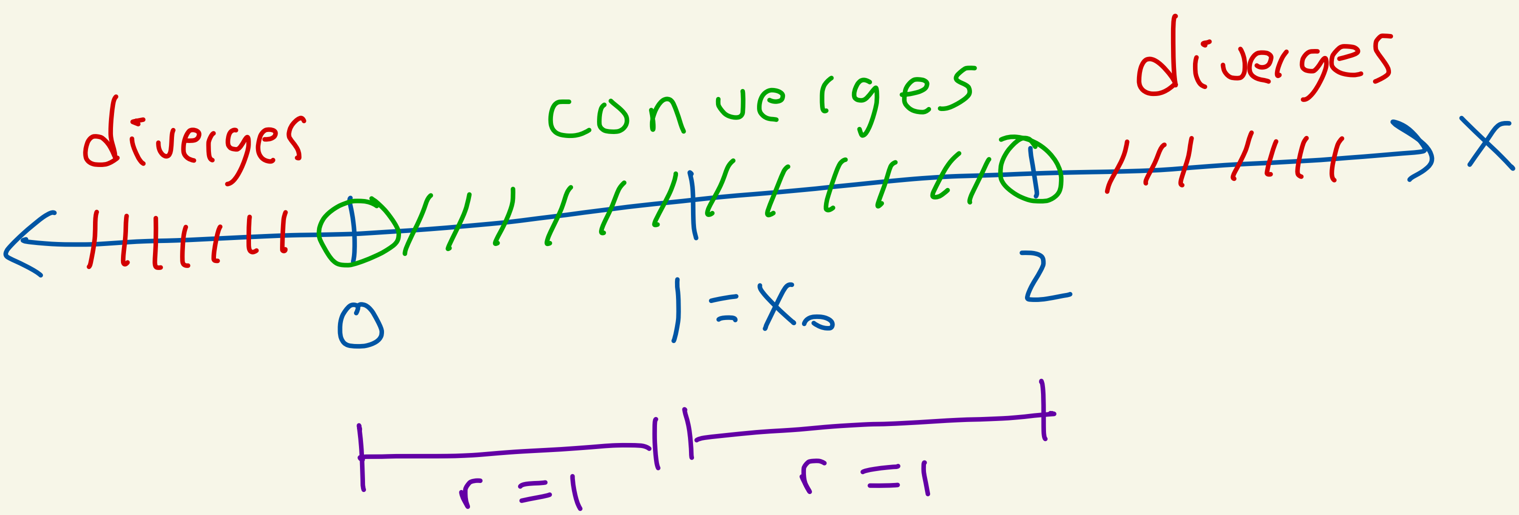
Ex:

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$



Let $0 < x < 2$,
Differentiate both sides of above

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$



Same radius $r = 1$, but endpoints not same convergence.